Minimal decoupling indices for linear systems

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Abstract: We show that for any right invertible linear system, there is a unique minimum list integers that represent the least expensive increases of infinite structure to produce for decoupling by non regular static state feedback, without changing the essential orders. In this case, this original list of minimal decoupling indices allows to establish necessary and sufficient conditions when the couplings between $\mathcal{R}^*$ and the rest of the system do not affect the part that must change the structure at infinity, which is, to our knowledge, a second original contribution.

Keywords: linear systems, non regular feedback, diagonal decoupling, structural properties.

1. INTRODUCTION

We consider the diagonal decoupling by state feedback for linear invariant systems. The regular case, which retains all the entries for the controlled system, was solved since 1971. A non-regular feedback needs less entries than the regular. It can lead to solutions whereas none exist in the regular case. With such controls all the invariant structures that make up the skeleton of a dynamic system can be modified and the question is: what new structure should we target and how can we obtain it? The non-regular diagonal decoupling was solved by dynamic state feedback in 1988 by Dion and Commault, exploiting new invariants, the essential orders, which are the minimal new infinite structure to reach for decoupling. The problem is to increase the infinite structure so that it coincides with the essential orders, knowing that they can always be kept by the dynamic feedbacks. This is no longer true in the static case. We consider here the Static Reduced Morgan's Problem (SRMP), say the static decoupling without increasing the essential orders. This particular case is in itself very interesting: it provides insight into the complex mechanisms of structural changes by non-regular controls. There are so far only very partial results for the SRMP: when it is suffisant to increase only one element of the infinite structure, Herrera and Lafay 1993, or for trivial internal structures, Zagalak et al 1998. The specific difficulties of SRMP, which were obscured in the dynamic case, are: firstly, the increases of infinite structure to solve SRMP depend on the order of the outputs of the system although the sum of these increases remains the same, secondly we must take into account internal unobservable couplings of the system from the outputs to be decoupled. For the first lock, we show that there exists an order of the outputs for which the increases of infinite structure are easier to achive, regarding the internal structure of the system. We give an algorithmic procedure to determine this unique list of "minimal decoupling indices". The second lock is not yet fully lifted, but we provide solutions to SRMP in less restrictive settings, to our knowledge, that what exists in the literature.

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2. PRELIMINARIES AND NOTATIONS

In all the sequel, Σ denotes a linear system whose state is supposed to be measured or reconstructible:

$$\Sigma \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t). \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}_n, u \in \mathcal{U} \subset \mathbb{R}_m$ and $y \in \mathcal{Y} \subset \mathbb{R}_p$ are the outputs to be controlled. Without loss of generality, we suppose B monic, C epic and Σ controllable, right invertible and without finite zeros. This last assumption implies that $\mathcal{V}^* = \mathcal{R}^*$, $\mathcal{V}^*$ and $\mathcal{R}^*$ denoting respectively the supremal $(A-B)$invariant and the supremal controllability subspaces contained in the kernel of C. Let us recall that the finite zeros are only involved when dealing with internal stability, which is not the case in this paper. A set of p elements is noted $\{\bullet\}_p$. A polynomial and a rational expression in the variable s are respectively noted $\bullet[s]$ and $\bullet(s)$. The degree of the polynomial $p[s]$ is noted $\partial p[s]$ and $\partial_{c_i} M[s]$ is the highest degree of polynomials of the i-th column of the polynomial matrix $M[s]$. Let us recall some classical elements of the polynomial approach, Kailath (1980):

Definition 1. A biproper rational $n \times n$ matrix $B(s)$ is a proper non singular matrix, the inverse of which is proper.

Property 1. The determinant of a biproper matrix $B(s)$ is a nonzero real constant when $s \to \infty$.

Property 2. Let a $n \times n$ biproper matrix $B(s)$ and a polynomial $n \times p$ matrix $M[s]$. Then: $\partial_{c_i} M[s] = \partial_{c_i} B(s)M[s]$, $i = 1, ..., p$.

This follows directly from Property 1: $B(s) = B_\infty + B_1(s)$, where $B_1(s)$ is strictly proper and $B_\infty = \lim_{s\to\infty} B(s)$ is a real invertible matrix.

Property 3. Let $B(s)$ a biproper $n \times n$ matrix with the structure: $B(s) = \begin{bmatrix} B_{1,1} & (0) \\ B_{2,1} & \mathbb{I}_k \end{bmatrix}$, where $\mathbb{I}_k$ is the $k \times k$ identity matrix. By Property 1 $B_{1,1}$ is a biproper matrix.

A fundamental link between regular state feedbacks $u(t) = Fx(t) + Gv(t)$, where F is a static or dynamic matrix and G is invertible, and biproper transformations on a transfer matrix is due to Hautus and Heymann (1978):

Property 4. A right biproper transformation $R(s)$ on the transfer matrix of Σ is realisable by a regular state feedback if and only if, for any polynomial entry u(s) such that $(sI_n - A)^{-1}Bu(s)$ is polynomial, $R^{-1}(s)u(s)$ is polynomial too.

This condition means that $R(s)$ does not add finite zeros. This is always verified by the biproper transformations used in the sequel. Note also that a right biproper operation on the transfer matrix is equivalent to a left biproper operation on it's inverse, particularly on the interactor defined in Wolovich and Falb (1976):

Definition 2. The interactor of Σ is the unique $p \times p$ triangular and non-singular polynomial matrix $\Phi[s] = [\varphi_{ij}[s]]$ such that there exists a biproper $m \times m$ (non unique) matrix $B_1(s)$ satisfying : $T(s) = C(sI_n - A)^{-1}B = \left[\Phi^{-1} 0\right] B_1(s)$ where:

- $\Phi_{ii} = s^{f_i}$, $i = 1, ..., p$, f_i being positive integers
- Φ_{ij} is zero or Φ_{ij}/s^{f_i} is divisible by s, $\forall j > i$.

This interactor depends on the order chosen for the outputs $y(t)$. It is unique under the action of the group (T, F, G), where T and G are changing of bases on $\mathcal{X}$ and $\mathcal{U}$, and F is a state feedback.
Four lists of integers characterize a part of the structure of Σ. For more details see Morse (1973),Commault et al. (1986), Cremer (1971):
- the controllability indices of (A, B) denoted $\{c_i\}_m$,
- the non-decreasing I_4 Morse's list $\{n'_i\}_p$, which consists of the orders of the zeroes at infinity of Σ,
- the non-decreasing I_2 Morse's list $\{\sigma_j\}_{m-p}$, characterizes the structure of $\mathcal{R}^*$. These integers are the controllability indices of the pair $(\mathcal{R}^* \mid (A + BF) \mid \mathcal{R}^*, \mathcal{B} \bigcap \mathcal{R}^*)$, where $(\mathcal{R}^* \mid (A + BF) \mid \mathcal{R}^*)$ is the double restriction of $(A+BF)$ to $\mathcal{R}^*$ and F is such that $(A + BF)\mathcal{V}^* \subset \mathcal{V}^*$,
- the essential orders $\{n_{ie}\}_p$ of the outputs of Σ: $n_{ie} = \partial_{c_i}\Phi[s]$, $i = 1, ..., p$, where $\Phi[s]$ is the interactor of Σ.

Property 5. (Herrera H et al. (1997)): These lists are invariant under the action of the group (T, F, G, Π), where Π is a permutation of the outputs of Σ.

Note that Π permutes the integers of $\{n_{ie}\}_p$ and that, in general, $I_2 = \{\sigma_j\}_{m-p}$ is not a sublist of $\{c_i\}_m$.

Property 6. Any system Σ satisfies:

$$\sum_{i=1}^{p} n'_i + \sum_{j=1}^{m-p} \sigma_j = n. \tag{2}$$

Let us recall some structural properties of the interactor of Σ, Lafay et al. (1992), Dion and Commault (1988):

Proposition 1. $\Phi[s] = W(s)diag(s^{n_{ie}})$, where $W(s)$ is a proper matrix of rank k called "proper part of the interactor." After a permutation (not unique) of the outputs of $T(s)$ the interactor of $T(s)\Pi$ has the structure:

$$[\varphi_{i,j}] = \begin{bmatrix} \Phi_1[s] & (0) \\ \Phi_2[s] & \Phi_3[s] \end{bmatrix}, \tag{3}$$

where:

- $\Phi_3[s] = diag(s^{n_{je}})$, the k integers n_{je} are extracted from the list of the essential orders of $\Phi[s]$
- The nonzero elements of the infinite structure of the proper part of the interactor are given by the list $\{\delta_i\}_{p-k}$, where $\delta_i = \partial\varphi_{i,i}$ for $i = 1, ..., p-k$.

Morse showed that the interactor provided the structure of the controllable and observable part of Σ, and this structural characterization has been extended to the non-observable part in Herrera H and Lafay (1993), to show off the structure of $\mathcal{R}^*$ and its coupling with the observable part of Σ:

Proposition 2. Given Σ, it is always possible to define $m - p$ fictitious outputs leading to an extended system, the $m \times m$ extended interactor of which has the structure:

$$\Phi_e[s] = \begin{bmatrix} \Phi_{1e}[s] & (0) \\ \Phi_{2e}[s] & \Phi_{3e}[s] \end{bmatrix}, \tag{4}$$

where $\Phi_{1e}[s]$ is the $p \times p$ interactor of Σ and $\Phi_{3e}[s] = diag\{s^{\sigma_1}, ..., s^{\sigma_{m-p}}\}$.

$\Phi_{1e}[s]$ may still be structured as in Proposition 3. To obtain $\Phi_e[s]$, just choose the $m \times n$ output matrix:

$$C_e = \begin{bmatrix} C & (0)_{p\times\sum_{j=1}^{m-p}\sigma_j} \\ (0)_{(m-p)\times\sum_{i=1}^{p} n'_i} & diag\{1\ 0\ ...\ 0.\}_{\sigma_j} \end{bmatrix}, \tag{5}$$

Remark 1. If $\Phi_{2e}[s] = 0$, the list $I_2 = \{\sigma_j\}_{m-p}$ corresponds to the controllability indices $\{c_j\}_{m-p}$ of the entries of $\mathcal{R}^*$. Generally, $\sigma_j \geq c_j$.

This follows directly from the "semi-canonical Morse's form" associated with $\Phi_e[s]$, Herrera H et al. (1997).

3. THE NON REGULAR DECOUPLING

The diagonal decoupling of Σ by state feedback, or Morgan's Problem, is as follows: under which conditions are there static (or dynamic) state feedbacks $u = Fx + Gv = Fx + [G_1G_2...G_p][v_1v_2...v_p]^T$ such that, for any $i \in p$, v_i controls the scalar output y_i without affecting the other outputs y_j? Let us recall some conditions of existence of regular static state feedbacks (G invertible) that solve this problem. The first results are in Morgan Jr (1964) and in Falb and Wolovich (1967). The regular problem is completely solved in Morse and Wonham (1971). We give here a structural version of this result, Commault et al. (1986):

Theorem 3. The following necessary and sufficient conditions are equivalent for the existence of a regular static state feedback which decouples Σ:

(i) The lists $\{n'_i\}_p$ and $\{n_{ie}\}_p$ are the same.
(i) The interactor of Σ is diagonal.

For the non-regular Morgan's problem, G is strictly monic. It was proved that, when Σ is not regularly decouplable, the smallest infinite structure to reach using a non regular control law is greater than or equal to the essential orders, Commault et al. (1986), Herrera H and Lafay (1993). This increase of structure can be performed using the integrators and the entries of $\mathcal{R}^*$ in the case of non-regular static state feedbacks, or by using external chains of integrators that will be controlled by entries of $\mathcal{R}^*$ in the case of non-regular dynamic state feedbacks. We address here the Static Reduced Morgan's Problem (SRMP): is it possible to find a non-regular static state feedback that allows to match the infinite structure of the closed loop system with the initial essential orders without changing these orders? The dynamic problem was solved in Dion and Commault (1988):

Theorem 4. The dynamic Morgan's Problem is solvable if and only if Σ is right invertible and $m - p \geq p - k$, k being the rank at infinity of the proper part of the interactor. In this case, a solution can always be obtained with $\sum_{i=1}^{p} n_{ie} - \sum_{i=1}^{p} n'_i$ integrators and the essential orders are not modified.

These conditions are necessary for SRMP. The dynamic solution can be directly explained from the structure of the interactor (3). We apply the following iterative procedure for $i = 1, 2, ..., p-k$:
- for $i = 1$, the entry u_1 will be replaced by an external chain of $h_1 = n_{1e} - \partial\varphi_{1,1}$ integrators to be controlled by an entry of $\mathcal{R}^*$, for instance $v_1 = u_{p+1}$. This amounts to multiplying the first row of (1) by s^{h_1}. So, the degree of the diagonal polynomial $\varphi_{1,1}$ becomes n_{1e} and it is possible, by means of a left biproper operation, to eliminate all other polynomials of the first column of (1).
- we successively made the same operation for $i = 2, 3, ..., p-k$, taking at each step a different entry of $\mathcal{R}^*$. Note that the integrators of $\mathcal{R}^*$ are never taken into account. The final interactor is given by the diagonal $p \times p$ matrix [diag $s^{n_{ie}}$], the list $\{n_{ie}\}_p$, being the essential orders arranged according to the outputs of the interactor (3). Thus, the structure at infinity is increased from $\{n'_i\}_p$ to $\{n_{ie}\}_p$ for the closed loop system, without the essential orders have been modified. The new system, with its entries $\{v_1, v_2, ..., v_{p-k}, u_{p-k+1}, ..., u_p\}$ is regularly decouplable from Theorem 3. In this procedure, we create $p-k$ independent chains of integrators of lengths $\{h_i\}_{p-k}$:

Definition 3. The "decoupling indices" of Σ are given by $\{h_i\}_{p-k}$.

Remark 2. This set depends on the order choosen for the outputs of Σ.

SRMP is much more complex. The $p-k$ independent chains of integrators $\{h_i\}_{p-k}$ must be generated from the $m-p$ chains of integrators of $\mathcal{R}^*$. The solution is based on the results of Loiseau (1988) on changing the structure at infinity of a linear system via non-regular state feedbacks:

Theorem 5. Consider a linear system for which $\{n'_i\}$ is the infinite structure and $\{\sigma_i\}$ is the I_2 Morse's list. Let $\{p_i\}$ a given list of integers. Note $\{v_i\}$, $\{\alpha_i\}$ and $\{\pi_i\}$ the dual lists of, respectively, $\{n'_i\}$, $\{\sigma_i\}$ and $\{p_i\}$. Let $\{\Gamma_i\}$ the list obtained by arranging the differences $(\pi_i - v_i)$ in a non increasing order. Then, there exists a static state feedback such that the structure at infinity of the closed loop system $\Sigma(C, A+BF, BG)$ is the list $\{p_i\}$ if and only if:

$$v_1 - v_i \geq \pi_1 - \pi_i, \forall i \geq 1, \tag{6}$$

$$\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \Gamma_i, \forall j \geq 1. \tag{7}$$

The solution of SRMP seems obvious. This is not, even if there is no coupling between $\mathcal{R}^*$ and the blocks of infinite structure corresponding to $\Phi_{2e}[s] = 0$ in (2).

Consider the decoupling indices $\{h_i\}_{p-k}$ and apply Theorem 5 taking for list $\{n'_i\}$ the structure at infinity $\{\partial\varphi_{i,i}\}_{p-k}$ of the proper part of (3), for list $\{\sigma_i\}_{m-p}$ the list I_2 of Σ and for list $\{p_i\}$ the list $\{n_{ie}\}_{p-k}$ of (3). Note that, as $\partial\varphi_{i,i} < n_{ie}$ for each column of Φ_2, conditions (6) of Theorem 5 are always verified.

This will not be satisfactory because, even if applying Theorem 5 the list $\{\partial\varphi_{i,i}\}$ will be led globally to $\{n_{ie}\}_{p-k}$, there is no guarantee that this will be done term by term, i.e. that $\partial\varphi_{i,i}$ becomes equal to n_{ie} for $i = 1, ..., p-k$. If this is not the case, the essential orders will change. To overcome this, it suffices to apply Theorem 5 choosing for list $\{n'_i\}_{p-k}$ the list $\{1\}_{p-k}$ and try to turn it into the list $\{1 + h_i\}_{p-k}$. This amounts to build, from the $m-p$ chains of integrators of lengths σ_i of $\mathcal{R}^*$, $p-k$ independent chains of lengths $\{h_i\}$ to replace the external chains used in the dynamic procedure. Note that condition (6) is still always true. So there remains only conditions (7) which become:

Proposition 6. SRMP have a solution for a system Σ such that $\Phi_{2e}[s] = 0$ if:

$$\sum_{j=1}^{i} \alpha_j \geq \sum_{j=1}^{i} \Gamma_j, \forall i \geq 1, \tag{8}$$

where $\{\Gamma_i\}_{\sup(h_i)}$ is the dual list of $\{h_i\}_{(p-k)}$ and $\{\alpha_i\}_{\sup(\sigma_i)}$ the dual list of $\{\sigma_i\}_{(m-p)}$.

This only leads to a sufficient condition because the list $\{h_i\}_{p-k}$ of decoupling indices depends on the order choosen for the outputs selected to obtain (3) although, overall, the sum of these indices is constant and equal to $\sum_{i=1}^{p-k} h_i = \sum_{i=1}^{p} n_{ie} - \sum_{i=1}^{p} n'_i$.
So we attach the following problem that Commault and Dion did not have had to solve the dynamic case: does it exists a (non necessarily unique) permutation of the outputs of Σ for which the list $\{h_i\}_{p-k}$ is such that, if the conditions of Proposition 6 are not checked for this list, then they will never be for another list $\left\{\hat{h}_i\right\}$ resulting from a different ordering of the outputs? We will prove that this unique list always exits and call it: list of "minimum decoupling indices" of Σ.
For this, we need the notion of "minor" list :

Definition 4. Let two lists of integers $\{\delta_i\}_{k1}$ and $\{\gamma_i\}_{k2}$ be such that $\sum_{i=1}^{k1} \delta_i = \sum_{i=1}^{k2} \gamma_i$. Note $\{\hat{\delta}_i\}_{sup\delta_i}$ and $\{\hat{\gamma}_i\}_{sup\gamma_i}$ their respective dual lists. The list $\{\delta_i\}_{k1}$ is a minor list if, for $i = 1, .., \sup(\sup_j(\delta_j), \sup_j(\gamma_j))$, we have:

$$\sum_{j=1}^{i} \hat{\delta}_j \leq \sum_{j=1}^{i} \hat{\gamma}_j. \tag{9}$$

.

For example, consider the two following lists of which the sum of the terms is the same: $L_1 = \{2, 4, 4\}$ and $L_2 = \{3, 3, 4\}$. Their respective dual lists are: $l_1 = \{3, 3, 2, 2\}$ and $l_2 = \{3, 3, 3, 1\}$. So L_1 is a minor list of L_2. If $L_2 = \{1, 1, 2, 6\}$, its dual list is $l_2 = \{4, 2, 1, 1, 1, 1\}$. L_1 is a minor list again.

Remark 3. : With regard to SRMP:

- as the rank at infinity k of the proper part of the interactor does not depend of the order of the outputs, the first of the conditions (8) requires that the list being sought contains $p-k$ terms,
- the sum of all terms of any candidate list for SRMP is the same and is equal to $\sum_{i=1}^{p} n_{ie} - \sum_{i=1}^{p} n'_i$.

4. MINIMUM DECOUPLING INDICES

In this section, it is suffisant to work with the interactor $\Phi[s]$ of Σ. We will proceed in two steps: first we will show that the choice made for the outputs on the part $\Phi_3[s]$ of (3) does not affect the smallest of these decoupling indices. Then we will develop an iterative procedure with $p-k-1$ stages and based on permutations of columns, which at each step, will provide us the smallest element to complete the non decreasing list of minimum decoupling indices. Let the interactor $\Phi[s] = \begin{bmatrix} \Phi_1[s] & (0) \\ \Phi_2[s] & \Phi_3[s] \end{bmatrix} = [\varphi_{i,j}]$, of (1) under the form (3). Thus, $\Phi_1[s]$ has the structure of a $p-k \times p-k$ interactor, and $\Phi_3[s] = diag(s^{n_{je}})$, for $j = p-k+1, ..., p$.
For columns $1, 2, ..., p-k$, we note p_{ie} the maximum degree of the polynomials of the i-th column $\Phi_1[s]$, $\delta_i = n_{ie} - p_{ie}$ and $\Delta^{p-k} = \min \{\delta_i\}_{p-k}$.

Definition 5. Δ^{p-k} is the smallest decoupling index of Σ.

According to the Proposition 1, the polynomial of greater degree of each column of $\begin{bmatrix} \Phi_1[s] \\ \Phi_2[s] \end{bmatrix}$ belongs to $\Phi_2[s]$. Let $\varphi_{t,j}$, with $j \leq p-k$ and $t > p-k$, this polynomial for the j-th column of $\Phi[s]$. Its degree is n_{je}. Let Π the permutation of the t-th and j-th columns of $\Phi[s]$, which corresponds to the permutation of the t-th and j-th outputs of (1). After this permutation, $\Phi[s]\Pi$ is no longer an interactor.
We will now determine the interactor $\Phi_\Pi[s]$ of $\Phi[s]\Pi$. To simplify the notations and without loss of generality, we assume that $j = 1$ and $t = p$. Noting (only in the following equation) $\alpha = p-k$, $\Phi[s]\Pi$ is given by:

$$\begin{bmatrix} 0 & 0 & . & . & . & . & 0 & \varphi_{1,1} \\ 0 & \varphi_{2,2} & . & . & . & . & 0 & \varphi_{2,1} \\ . & . & . & . & . & . & . & . \\ 0 & \varphi_{\alpha,2} & . & \varphi_{\alpha,\alpha} & . & . & 0 & . \\ . & . & . & . & \varphi_{\alpha+1,\alpha+1} & . & . & . \\ . & . & . & . & 0 & . & . & . \\ 0 & \varphi_{\alpha,\beta} & . & . & 0 & 0 & \varphi_{p-1,p-1} & . \\ \varphi_{p,p} & \varphi_{p,2} & . & \varphi_{p,\alpha} & 0 & 0 & 0 & \varphi_{p,1} \end{bmatrix} \tag{10}$$

We will determine a biproper matrix $B_\Pi(s)$ such that $\Phi_\Pi[s] = B_\Pi(s)\Phi[s]\Pi$ is the interactor of (10). As a first step, we cancel by a first left biproper operation $B_1(s)$ the polynomials $\varphi_{j,1}$, $j = 1, ..., p-1$, of the p-th column of (10). Note that $B_1(s)$ always exists since, by hypothesis, $\partial\varphi_{p,1}[s] = n_{1e} \geq \partial\varphi_{j,1}[s]$, j=1,...,p-1. Choose:

$$B_1(s) = \begin{bmatrix} 1 & 0 & . & . & . & . & . & 0 & -\frac{\varphi_{1,1}}{\varphi_{p,1}} \\ 0 & 1 & 0 & . & . & . & . & 0 & -\frac{\varphi_{2,1}}{\varphi_{p,1}} \\ . & & . & & & & & & . \\ . & & & & & & & & . \\ 0 & . & . & . & . & . & 0 & 1 & -\frac{\varphi_{p-1,1}}{\varphi_{p,1}} \\ 0 & 0 & . & . & . & . & . & 0 & 1 \end{bmatrix}. \tag{11}$$

Then:

$$B_1(s)\Phi\Pi = \begin{bmatrix} \hat{\Phi}_1[s] & (0) \\ \hat{\Phi}_2[s] & \hat{\Phi}_3[s] \end{bmatrix} = [\hat{\varphi}_{i,j}], \tag{12}$$

where $\hat{\Phi}_1[s]$ is a $p-k \times p-k$ matrix, and where $\hat{\Phi}_3[s]$ and $\Phi_3[s]$ differ only by the polynomial $\hat{\varphi}_{p,p} = \varphi_{p,1}$ while $\hat{\varphi}_{p,1} = s^{n_{pe}} = \varphi_{p,p}$. The other polynomials of the p-th row of $B_1(s)\Phi\Pi$ are not modified [1].

Lemma 7. The position of polynomials of higher degree of columns of $\hat{\Phi}_2[s]$ and $\Phi_2[s]$ is not changed by the transformation described above, and this higher degree is equal to n_{ie} for $i = 2, 3, ..., p-k$.

Proof. As $B_1(s)$ is a biproper matrix, Property 2 implies that the maximum degrees of each column of $\Phi[s]\Pi$ and $B_1(s)\Phi[s]\Pi$ are the same.
Let $\varphi_{j,m}$ this polynomial of maximum degree for the m-th column of $\Phi[s]$, with $p-k+1 \leqslant j \leqslant p-1$ and $2 \leqslant m \leqslant p-k$. From the definition of the essential orders, $\partial\varphi_{j,m} = n_{me}$. In equation (12), $\varphi_{j,m}$ is transformed as $\hat{\varphi}_{j,m} = \varphi_{j,m} - \frac{\varphi_{p,m}}{\varphi_{p,1}}\varphi_{j,1}$. As $\partial\varphi_{p,m} < \partial\varphi_{j,m}$ and $\partial\varphi_{p,1} > \partial\varphi_{j,1}$ we have $\partial\varphi_{j,m} > \partial\frac{\varphi_{p,m}}{\varphi_{p,1}}\varphi_{j,1}$ and then $\hat{\varphi}_{j,m}$ remains the polynomial of maximun degree of the m-th column of $\hat{\Phi}_2[s]$.

Consider now $\hat{\Phi}_1[s]$. We will prove the following Lemma:

Lemma 8. : The index Δ^{p-k} is the same for $\Phi[s]$ and for $\Phi_\Pi[s]$.

Proof. 1 - At the step corresponding to equation (12), each polynomial $\hat{\varphi}_{i,1}$ of the first column of $\hat{\Phi}_1[s]$ is given by:

$$\hat{\varphi}_{i,1} = -\varphi_{p,p}\frac{\varphi_{i,1}}{\varphi_{p,1}}, \tag{13}$$

with $\partial\varphi_{p,p} = n_{pe}$ and $\partial\varphi_{p,1} = n_{1e}$. Then $n_{pe} - \partial\hat{\varphi}_{i,j} = n_{1e} - \partial\varphi_{i,j}$. So $\hat{\delta}_1 = \delta_1$. Especially, if we had $\delta_1 = \Delta^{p-k}$, we have kept this index in the first column. But still nothing proves here that $\hat{\delta}_1 = \min\left\{\hat{\delta}_i\right\}_{p-k}$.

2 - For the other columns, each polynomial $\varphi_{i,j}$ of $\Phi_1[s]$ is turned in:

$$\hat{\varphi}_{i,j} = \varphi_{i,j} - \varphi_{p,j}\frac{\varphi_{i,1}}{\varphi_{p,1}}, j = 2, ..., p-1 \tag{14}$$

with $\partial\varphi_{p,1} = n_{1e}$. So we have:

$$\partial\hat{\varphi}_{i,j} \leq \max(\partial\varphi_{i,j}, \partial\varphi_{p,j}\frac{\varphi_{i,1}}{\varphi_{p,1}}). \tag{15}$$

Then:
$\partial\hat{\varphi}_{i,j} \leq \max(\partial\varphi_{i,j}, \partial\varphi_{p,j}\frac{\varphi_{i,1}}{\varphi_{p,1}})$.
So $\hat{\delta}_j = \min_i \{n_{je} - \partial\hat{\varphi}_{i,j}\}_{p-k} =$

[1] We could normalize $\varphi_{p,1}$ as $s^{n_{1e}}$, but this does not change anything at this level, as this would not affect the degrees of these polynomials and we will work only on the properties of degree

$\min_i \{n_{je} - \partial\varphi_{i,j}, n_{je} - \partial\varphi_{p,j} - \partial\varphi_{i,1} + \partial\varphi_{p,1}\}_{p-k}$.
Now, $\partial\varphi_{p,1} - \partial\varphi_{i,1} \geq \delta_1$, implies:
$n_{je} - \partial\hat{\varphi}_{i,j} \geq n_{je} - \partial\varphi_{p,j} + \delta_1 = \delta_1 + c$, where $c \geq 0$. Otherwise, $\min_i \{n_{je} - \partial\varphi_{i,j}\}_{p-k} = \delta_j$. So, $\hat{\delta}_j \geq \min\{\delta_j, \delta_1 + c\}$.
If $\delta_j < \delta_1$, $\hat{\delta}_j = \delta_j$ and in particular if $\delta_j = \Delta^{p-k}$ and $\delta_1 > \Delta^{p-k}$ we have $\hat{\Delta}^{p-k} = \Delta^{p-k}$. If $\delta_j = \Delta^{p-k} = \delta_1$, there may be cancellation of the terms of highest degree of the polynomial and then $\hat{\delta}_j > \Delta^{p-k}$. But, in this case, Δ^{p-k} is still in the first column, as we have seen in item 1.
The consequence is that, for each column j, $\hat{\delta}_j \geq \min(\delta_j, \delta_1)$. So $\hat{\delta}_j \geq \Delta^{p-k}$,with equality if $\delta_j = \Delta^{p-k} < \delta_1$, the case $\delta_j = \Delta^{p-k} = \delta_1$ being treated in item 1. This allows us to conclude the proof as follows: the interactor of $\Phi[s]\Pi$ will be obtained by the action of a second left biproper transformation $B_2(s)$ so that $B_2(s)B_1(s)\Phi[s]\Pi$ is an interactor. From (12), $B_2(s)$ has the following structure:

$$B_2(s) = \begin{bmatrix} B_{2,1}(s) & (0) \\ B_{2,2}(s) & \mathbb{I}_k \end{bmatrix}, \tag{16}$$

where, from Property 3, $B_{2,1}(s)$ is a biproper $p-k \times p-k$ matrix such that $B_{2,1}(s)\hat{\Phi}_1[s]$ has the structure of an interactor.

As $B_2(s)$ is biproper, the maximum degree of the polynomials of each column of $B_2(s)B_1(s)\Phi\Pi$ is equal to the essential order of the corresponding output and the polynomial(s) with this degree are in the same position(in the last k rows).
Moreover, as $B_{2,1}(s)$ is also biproper, the maximal degree of each column of $B_{2,1}(s)\hat{\Phi}_1[s]$ and of $\hat{\Phi}_1[s]$ are not modified and the differences $\hat{\delta}_i$ are kept for $\Phi_\Pi[s]$. Thus Δ^{p-k} is the same for $\Phi[s]$ and for $\Phi_\Pi[s]$.
Remark 4. It is important to note that the differences δ_i can only be unmodified or higher than $\delta_1 = \Delta^{p-k}$.

It is now possible to characterize a unique new list of integers, namely the minimum decoupling indices, which plays a key rule to solve the SRMP.

Theorem 9. There exists an unique list of positive integers $\{\Delta^i\}_{(p-k)}$, where k is the rank at infinity of the interactor (cf Proposition 3) such that:

- $\Delta^1 \geq \Delta^2 \geq ... \geq \Delta^{p-k}$,
- This list $\{\Delta^i\}_{(p-k)}$ is the minimal list of the decoupling indices, in the sense that if the conditions of Proposition 6 are not satisfied for this list, they will not be satisfied for any other list of decoupling indices $\{h_i\}$ resulting from a different ordering of the outputs of Σ.

Proof. Let the interactor $\Phi_{\Pi[s]} = B_2(s)B_1(s)\Phi\Pi$. If $\delta_{p-k} > \Delta^{p-k}$, there is at least one column of $\Phi_{\Pi[s]}$, e.g. the r-th, for which $\delta_r = \Delta^{p-k}$. Permute it with the $(p-k)$-th column. If necessary, we do, after this permutation of columns, a permutation of rows to place the polynomial having the maximum degree of the new $(p-k)$-th column in the position $p-k \times p-k$. We can then compute the new interactor $\tilde{\Phi}[s] = [\tilde{\varphi}_{i,j}]$ using the same procedure we have developed to prove Lemma8. The $(p-k)$-th column of this new interactor is such that $n_{(p-k)e} - \partial\tilde{\varphi}_{p-k,p-k} = \Delta^{p-k}$. Note then

$$\tilde{\Phi}[s) = \begin{bmatrix} \tilde{\Phi}_1[s] & (0) \\ \tilde{\Phi}_2[s] & \tilde{\Phi}_3[s] \end{bmatrix} = [\hat{\varphi}_{i,j}]\,. \tag{17}$$

$\tilde{\Phi}_1[s]$ has the structure of a $p-k-1 \times p-k-1$ interactor, and, for the columns $1, 2, ..., p-k-1$, we will note p_{ie} the maximun degree of the polynomials of the i-th column of $\tilde{\Phi}_1[s]$, $\tilde{\delta}_i = n_{ie} - p_{ie}$ and $\Delta^{p-k-1} = \min\left\{\tilde{\delta}_i\right\}_{p-k-1}$. Note that $\tilde{\delta}_i \geq \hat{\delta}_i$. Thus $\Delta^{p-k-1} \geq \Delta^{p-k}$. We then permute a column for which $\tilde{\delta}_i = \Delta^{p-k-1}$ with the $(p-k-1)$-th column of $\tilde{\Phi}[s]$ and we compute the new interactor. $\tilde{\Phi}_3[s]$ is not modified in this operation. This procedure is iterated until obtaining the unique list of the $p-k$ integers

$$\Delta^1 \geq \Delta^2 \geq ... \geq \Delta^{p-k} = \{\Delta^i\}_{p-k}. \tag{18}$$

Proposition 10. The final interactor

$$\bar{\Phi}[s] = \bar{B}[s]\Phi[s]\bar{\Pi} = [\bar{\varphi}_{i,j}] \tag{19}$$

is such that: first, for $i = 1, 2, ..., p-k$, $\bar{\varphi}_{i,i} = s^{n_{ie}-\Delta^i}$ and $\bar{\varphi}_{i,i}$ divides $\bar{\varphi}_{j,i}$, $j \geq i$ and secondly, for $i = 1, 2, ..., p-k$, $\bar{\varphi}_{i,i} = s^{n_{ie}}$

Here the essential orders n_{ie} are those of the initial system, but their respective places in the list has evolved with the successive output permutations represented by $\bar{\Pi}$. the biproper matrix $\bar{B}[s]$ is the product of all the left biproper transformations.

5. APPLICATION TO SRMP

Let $\bar{\Phi}_e[s]$ the extended interactor of Σ(Proposition 2):

$$\bar{\Phi}_e[s] = \bar{B}_e[s]\Phi[s]\bar{\Pi}_e = \begin{bmatrix} \bar{\Phi}_{1e}[s] & (0) \\ \bar{\Phi}_{2e}[s] & \bar{\Phi}_{3e}[s] \end{bmatrix} \tag{20}$$

with

$$\bar{B}_e[s] = \begin{bmatrix} \bar{B}[s] & (0) \\ (0) & \mathbb{I}_{m-p} \end{bmatrix}, \bar{\Pi}_e[s] = \begin{bmatrix} \bar{\Pi} \\ (0)_{m\times(m-p)} \end{bmatrix} \tag{21}$$

Through the permutation of the outputs described in section 4, the $p \times p$ interactor $\bar{\Phi}_{1e}[s]$ of Σ, is such that:

- $nie - \partial\bar{\varphi}_{i,i} = \Delta^i$, for $i = 1, 2, ..., p-k$, the list $\{\Delta^i\}_{p-k}$ being the list of minimum decoupling indices,
- $\bar{\varphi}_{j,j} = s^{n_{je}}$, $j = p-k+1, ..., p$, where n_{je} were arranged in non decreasing order,
- for $i \geq j$ and $i = p-k+1, ..., p$, $\bar{\varphi}_{i,j} = 0$,

and $\bar{\Phi}_{3e}[s] = diag\{s^{\sigma_i}\}_{m-p}$ where the integers σ_i are arranged in non decreasing order. We can write:

$$\bar{\Phi}_{2e}[s] = \left[\bar{\Phi}_{2,1}[s]\ \bar{\Phi}_{2,2}[s]\right], \tag{22}$$

where $\bar{\Phi}_{2,1}[s]$ is a $m-p \times p-k$ polynomial matrix. From the semi canonical Morse's form, Herrera H et al. (1997), the polynomials of $\bar{\Phi}_{2,2}[s]$ are zero or verify: $n_{je} + 1 \leq \partial\bar{\varphi}_{i,j} \leq \sigma_i$. Suppose in a first case that $\bar{\Phi}_{2e}[s] = 0$. Then, the integers σ_i are the controllability indices of the entries of $\mathcal{R}^*$ and the solution to the SRMP is given by:

Theorem 11. The SRMP admits a solution for Σ when $\bar{\Phi}_{2e}[s] = 0$ if and only if:

$$\sum_{j=1}^{i} \alpha_j \geq \sum_{j=1}^{i} \gamma_j, \tag{23}$$

where $\{\gamma_i\}_{\Delta^1}$ is the dual list of the list of minimum decoupling indices $\{\Delta^i\}_{(p-k)}$ and $\{\alpha_i\}_{\sup(\sigma_i)}$ is the dual list of the list $\{\sigma_i\}_{(m-p)}$.

Proof. The inverse of the interactor (20) is obtained after a permutation of the outputs and a right biproper operation on the transfert matrix of Σ. By Property 4, these tranformations does not change the decouplability of the system. So we can always suppose that the interactor has the form (20). The sufficienty follows directly from Proposition 6. The necessity comes from the fact that the list $\{\Delta^i\}_{(p-k)}$ minores, in the sense of Theorem 9, all the possible lists of decoupling indices for SRMP.

The list $\{\Delta^i\}_{(p-k)}$ does not appear explicitly or structurally in Zagalak et al. (1998).

Now consider the case where, in (22), the $(m-p) \times k$ block $\bar{\Phi}_{2,2}$, is not zero. Note $\{r_i\}_{(m-p)}$, $i = 1, ..., m-p$ the controllability indices of the entries of $\mathcal{R}^*$ when we consider the system Σ with only the entries $u_{p-k+1}, ..., u_m$ associated with (20). These indices correspond to the maximum lengths of the chains of integrators that can be used without increasing the essential orders of the outputs $y_{p-k+1}, ..., y_p$, Herrera H and Lafay (1993). So, if $\bar{\Phi}_{2,2}$ is not zero and $\bar{\Phi}_{2,1} = 0$, we have:

Theorem 12. The SRMP has a solution for a system Σ where $\bar{\Phi}_{2,1} = 0$ if and only if:

$$\sum_{j=1}^{i} \hat{\alpha}_j \geq \sum_{j=1}^{i} \gamma_j, \tag{24}$$

where $\{\gamma_i\}_{\Delta^1}$ is the dual list of the minimal list of decoupling indices $\{\Delta^i\}_{(p-k)}$ and $\{\hat{\alpha}_i\}_{\sup r_i}$ the dual list of the list $\{r_i\}_{(m-p)}$.

This result is, to my knowledge, original, but it remains a special case by imposing $\bar{\Phi}_{2,1}$ to be zero.

We can now apply the same procedure for any $\bar{\Phi}_2$, taking for the list $\{\hat{\sigma}_i\}_{(m-p)}$ the list of controllability indices of the entries of $\mathcal{R}^*$ considering all the entries of Σ. The list $\{\hat{\sigma}_i\}_{(m-p)}$ is now a sub-list of $\{c_i\}_m$.

Corollary 13. SRMP has a solution for a system Σ if:

$$\sum_{j=1}^{i} \hat{\alpha}_j \geq \sum_{j=1}^{i} \gamma_j, \tag{25}$$

where $\{\gamma_i\}_{\Delta^1}$ is the dual list of the minimal list of decoupling indices $\{\Delta^i\}_{(p-k)}$ and $\{\hat{\alpha}_i\}_{\sup r_i}$ the dual list of the list of controllability indices of the entries of $\mathcal{R}^*$.

This gives only a sufficient condition for SRMP because it is then necessary to know how the chains of integrators extraded from $\mathcal{R}^*$ are composed to achieve increases of infinite structure.

6. CONCLUSION

We have shown that for a linear right invertible system there exists a unique list for the lengths of the chains of integrators necessary to solve SRMP such that if there is no solution for this list, no other list may be satisfactory. This leads to necessary and sufficient conditions for SRMP in some particular cases, one of which is original. May be, it lacks today structural information(s) to solve the problem in the general case. This becomes a very complex problem including a delicate combinatorial aspect.

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